Lecture 3
Wake fields

January 24, 2019

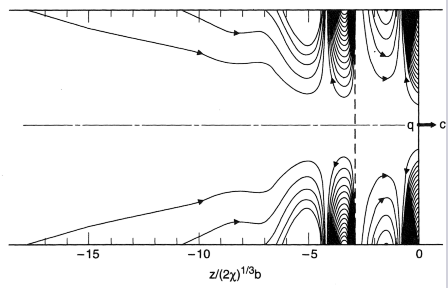
Lecture outline

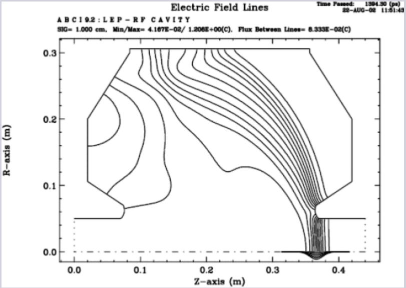
- Longitudinal and transverse wakes—definitions and properties.
- Panofsky-Wenzel theorem.
- Wakes in axisymmetric geometry.
- Wakes in system with planes of symmetry.
- Resonator wake.

We discussed in the previous lecture that the electromagnetic interaction between particles of a relativistic beam moving in free space is suppressed ($\propto 1/\gamma^2$). In practice, such interaction is often determined by the presence of material walls of the vacuum chamber and occurs if 1) the pipe is not cylindrical (which is usually due to the presence of RF cavities, flanges, bellows, beam position monitors, slots, etc., in the vacuum chamber), or 2) the wall of the chamber is not perfectly conducting.

Effect on environment on the beam

In a vacuum chamber the electromagnetic field that the beam feels is different from the vacuum field. See illustrations.





The fields can drive beam instabilities in a ring or deteriorate beam properties (e.g., the projected emittance or the energy spread) in linacs. We want to characterize the forces exerted by these fields on charges in a general setup. After we know the forces, we will be able to study the beam dynamics with account of these forces.

Assumptions

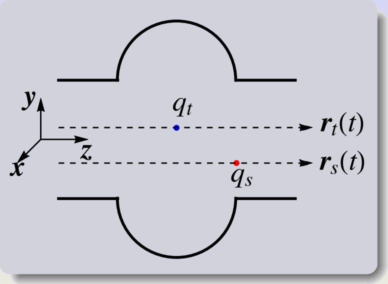
In the classical wake field theory we make several simplifying assumptions/approximations.

- We use the linearity of Maxwell's equations and boundary conditions — the force is proportional to the beam charge.
- When we calculate the forces, we assume that the beam is moving along a *straight* line through the region that generates a wake (that is we neglect the orbit curvature in this region). The electromagnetic interaction of charged particles in accelerators with surrounding environment is usually a relatively small effect that can be considered as a perturbation.
- We characterize the effect of the wake field forces by their integrated along the orbit values.
- For relativistic beams we assume $v = c$. This introduces the *causality* principle for the wakes.

These assumptions are typically well satisfied in reality (when one deals with relativistic beams), although there are special cases when each of them can be violated.

Definition of wake, case 1

1. A localized source of the wake (a cavity, a flange, a bellow, ...).



A *source particle* q_s passing through a localized obstacle in the beam pipe creates fields $\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{B}(\boldsymbol{r}, t)$ that act on the *test particle* q_t,

$$\boldsymbol{r}_s(t) = x_s\hat{\boldsymbol{x}} + y_s\hat{\boldsymbol{y}} + ct\hat{\boldsymbol{z}}$$
$$\boldsymbol{r}_t(t) = x_t\hat{\boldsymbol{x}} + y_t\hat{\boldsymbol{y}} + (ct - s)\hat{\boldsymbol{z}}$$

For $s > 0$, the test particle is behind the source particle.

Calculate the change of the momentum $\Delta\boldsymbol{p}_t$ of the test particle caused by the fields generated by the source particle,

$$\Delta\boldsymbol{p}_t = q_t \int_{-\infty}^{\infty} dt \, [\boldsymbol{E}(\boldsymbol{r}_t(t), t) + c\hat{\boldsymbol{z}} \times \boldsymbol{B}(\boldsymbol{r}_t(t), t)]$$

We assume that the integrals converge at infinity. The wake:

$$\boldsymbol{w}(x_s, y_s, x_t, y_t, s) = \frac{c}{q_s q_t}\Delta\boldsymbol{p}_t$$

Longitudinal and transverse wakes

We usually separate the longitudinal and transverse components of $\boldsymbol{w}$ introducing the *longitudinal* and *transverse wake functions*

$$w_\ell = -\frac{c}{q_s q_t}\Delta p_{t,z} = -\frac{c}{q_s}\int dt\, E_z(\boldsymbol{r}_t(t), t), \tag{3.1}$$
$$\boldsymbol{w}_t = \frac{c}{q_s q_t}\Delta \boldsymbol{p}_\perp = \frac{c}{q_s}\int dt\, [\boldsymbol{E}_\perp(\boldsymbol{r}_t(t), t) + c\hat{\boldsymbol{z}} \times \boldsymbol{B}(\boldsymbol{r}_t(t), t)]$$

Note the minus sign in the definition of w_ℓ — a positive wake means an energy loss of the test particle (assuming $q_s q_t > 0$). The longitudinal wake has the dimension of energy/charge^2 (e.g., V/C=Ω/s). In CGS units, the wake dimension is cm^{-1} ($1\ \mathrm{V/pC} = 1.11\ \mathrm{cm}^{-1}$).

Wake field and impedance calculations are often done in CGS units. To convert from cgs to MKS simply set:

$$\frac{Z_0 c}{4\pi} = 1$$

Definition of wake, case 2

2. The source of the wake is uniformly (or periodically) distributed along the path (the case of a resistive wall, periodic accelerating structure, etc.).
In a steady state (far from the entrance to the structure) the fields do not depend on z. It is more convenient to introduce the *wake per unit length* of the path

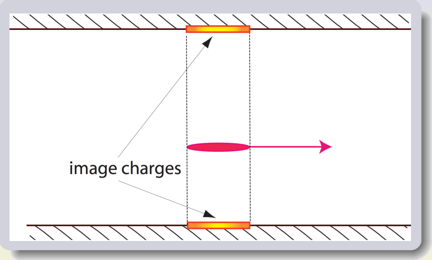
$$w_\ell(x_s, y_s, x_t, y_t, s) = -\frac{1}{q_s} E_z(\boldsymbol{r}_t(t), t), \tag{3.2}$$

$$\boldsymbol{w}_t(x_s, y_s, x_t, y_t, s) = \frac{1}{q_s} \left[\boldsymbol{E}_\perp(\boldsymbol{r}_t(t), t) + c\hat{\boldsymbol{z}} \times \boldsymbol{B}(\boldsymbol{r}_t(t), t)\right]$$

In a longitudinally uniform system the RHS here does not depend on time. [In a periodic structure one averages the fields over the structure period, $\langle \dots \rangle$.]
In this definition, the wakes acquire an additional dimension of inverse length, and have the dimension V/C/m in MKS (and cm^{-2} in CGS).
In case of periodic structure we average RHS over the period.

Particles moving in a perfectly conducting pipe

If particles move in a perfectly conducting cylindrical pipe of arbitrary cross section parallel to the axes, they induce image charges on the surface of the wall. The image charges travel with the same velocity v. Since the particles and image charges move on parallel paths, in the limit $v = c$, they do not interact with each other. Hence, $\boldsymbol{w} = 0$.



Mathematically, the boundary condition for the fields on the surface of a perfectly conducting metal is

$$\boldsymbol{E}_t = 0 \tag{3.3}$$

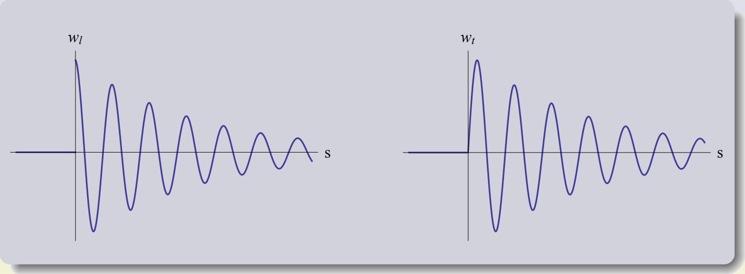
See the discussion of the Leontovich boundary condition in L5.

The interaction between the particles in ultrarelativistic limit can occur if 1) the wall is not perfectly conducting, or 2) the pipe is not cylindrical (due to the presence of RF cavities, flanges, bellows, beam position monitors, slots, etc., in the vacuum chamber), or 3) we drop the assumption $v = c$ (the space charge impedance).

General properties of wake functions

Since we assume that $v = c$ the wake does not propagate ahead of the leading charge, hence

$$w_\ell(x_s, y_s, x_t, y_t, s) = 0, \qquad \boldsymbol{w}_t(x_s, y_s, x_t, y_t, s) = 0, \qquad \text{for } s < 0\,.$$



[Exceptions: the space charge wake, the CSR wake.]

Panofsky-Wenzel theorem

The longitudinal and transverse wakes are not independent of each other, they are linked through the *Panofsky-Wenzel theorem*. It states that

$$\nabla^{(t)} \times \boldsymbol{w} = 0$$

where $\nabla^{(t)} = (\hat{\boldsymbol{x}}\partial/\partial x_t, \hat{\boldsymbol{y}}\partial/\partial y_t, \hat{\boldsymbol{\ell}}\partial/\partial s)$. It follows from this equation that

$$\frac{\partial \boldsymbol{w}_t}{\partial s} = \nabla_{\perp}^{(t)} w_\ell \equiv \hat{\boldsymbol{x}}\frac{\partial w_\ell}{\partial x_t} + \hat{\boldsymbol{y}}\frac{\partial w_\ell}{\partial y_t} \tag{3.4}$$

From $\nabla^{(t)} \times \boldsymbol{w} = 0$ it also follows that both wakes can be expressed in terms of a single function $V(x_s, y_s, x_t, y_t, s)$

$$w_\ell = \frac{\partial V}{\partial s}, \qquad \boldsymbol{w}_t = \hat{\boldsymbol{x}}\frac{\partial V}{\partial x_t} + \hat{\boldsymbol{y}}\frac{\partial V}{\partial y_t} \tag{3.5}$$

Panofsky-Wenzel theorem

Let's check that P-W holds,

$$\left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x_t} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y_t}\right) w_\ell = \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x_t} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y_t}\right)\frac{\partial V}{\partial s}$$
$$= \frac{\partial}{\partial s}\left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x_t} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y_t}\right) V = \frac{\partial}{\partial s}\boldsymbol{w}_t \tag{3.6}$$

Sometimes V is called the wake potential.
One can also prove the following property of the wake function

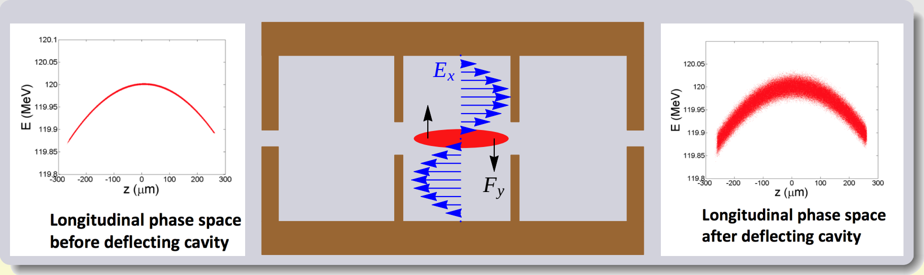
$$\nabla^{(t)} \cdot \boldsymbol{w}_t = 0$$

From this equation it follows that V satisfies the Laplace equation

$$\frac{\partial^2 V}{\partial x_t^2} + \frac{\partial^2 V}{\partial y_t^2} = 0 \tag{3.7}$$

1. ***E*** and ***B*** are not necessarily the fields of the first particle—they can also be an external electromagnetic field (but no charges and currents along the orbit of the test charge). In this case the Panofsy-Wenzel theorem is applicable as well.

Example: a transverse deflecting structure (TDS). [5]. TM_{100} mode is excited in the cavity.

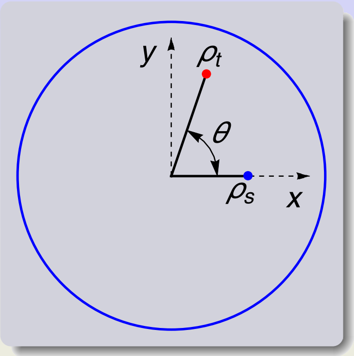


Longitudinal phase space before deflecting cavity

Longitudinal phase space after deflecting cavity

[5] C. Behrens et al., PRST-AB **15**, 022802 (2012).

Wakes in axisymmetric systems, multipoles



In an axisymmetric system V depends only on the absolute values of $\rho_s = \sqrt{x_s^2 + y_s^2}$, $\rho_t = \sqrt{x_t^2 + y_t^2}$, and the angle θ between them. Chose coordinate system so that $y_s = 0$. $V(\rho_s, \rho_t, \theta, s)$ will be a periodic even function of angle θ in the cylindrical coordinate system shown in the figure.

$$V(\rho_s, \rho_t, \theta, s) = \sum_{m=0}^{\infty} V_m(\rho_s, \rho_t, s) \cos(m\theta)$$

This is an expansion in the series of *multipoles*: $m = 0$ is the *monopole* wake, $m = 1$ is the *dipole*, $m = 2$ is the *quadrupole*, etc. [V_m are the main objects in A. Chao's textbook.]

Wakes in axisymmetric systems, expansion in powers of ρ

It is possible to find a general form of the dependence of W_m versus ρ_s and ρ_t using Maxwell's equations[6]:

$$V_m(\rho_s, \rho_t, s) = F_m(s)\rho_s^m \rho_t^m$$

We then have

$$w_\ell = \sum w_\ell^{(m)}, \qquad \boldsymbol{w}_t = \sum \boldsymbol{w}_t^{(m)}$$

were

$$\begin{aligned} w_\ell^{(m)} &= \cos(m\theta)\frac{\partial V_m}{\partial s} = \rho_s^m \rho_t^m F_m'(s)\cos(m\theta), \\ \boldsymbol{w}_t^{(m)} &= \nabla_\perp^{(t)} V_m \cos(m\theta) = m\rho_s^m \rho_t^{m-1} F_m(s)\left[\hat{\boldsymbol{\rho}}\cos(m\theta) - \hat{\boldsymbol{\theta}}\sin(m\theta)\right] \end{aligned}$$

where $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\theta}}$ are the unit vectors in the radial and azimuthal directions.

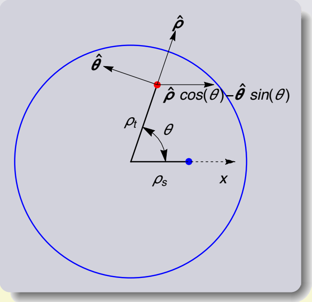
[6] The dependence $V_m \propto \rho_t^m$ follows from Eq. (3.7); however it is not easy to prove that $V_m \propto \rho_s^m$.

Wakes in axisymmetric systems, small offsets

For *small offsets*, near the axis, $m \geq 2$ terms can usually be neglected. One then keeps $m = 0$ (*monopole*) and $m = 1$ (*dipole*) wakes. For the monopole wake

$$w_\ell \equiv w_\ell^{(0)} = F_0'(s) = V_0'(s)$$

The monopole transverse wake vanishes, $w_t^{(0)} = 0$.



For the dipole wake ($m = 1$), the vector $\hat{\boldsymbol{\rho}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$ is directed along the x axis, that is in the direction of $\boldsymbol{\rho}_s = x_s\hat{\boldsymbol{x}} + y_s\hat{\boldsymbol{y}}$. Hence,

$$\boldsymbol{w}_t^{(1)} = \boldsymbol{\rho}_s F_1(s).$$

The dipole wake does not depend on the offset of the trailing particle! This is only true in axisymmetric systems.

Wakes in axisymmetric systems

For small offsets, the transverse wake is typically normalized by ρ_s, and $w_t^{(1)}/\rho_s$ is called the transverse dipole wake, $\bar{w}_t$, (we will use the over-bar notation for this wake),

$$\bar{w}_t(s) = F_1(s) \tag{3.8}$$

$\bar{w}_t$ has dimension of V/C/m or cm^{-2}. This $\bar{w}_t$ is what is usually called the transverse wake in accelerator literature; it should not be confused with $\boldsymbol{w}_t$ in (3.1)!

A positive transverse wake means the kick in the direction of the offset of the driving particle (if both particles have the same charge).

General properties of wakes

There is a number of general properties of the wake functions, see A. Chao's book. Classical wakes are localized behind the driving particle. Also, $|w_\ell(s)| \le w_\ell(0)$.

An important property of the wake is:

$$\int_0^\infty w_\ell(s) ds = 0 \tag{3.9}$$

We will return to this property later. If this is not satisfied for your model of wake, it means that the model is not valid for $0 < s < \infty$, but only on some finite interval of coordinate s.

Transverse wakes in non-axisymmetric systems

Assume a vacuum chamber with two perpendicular planes of symmetry: $x = 0$ and $y = 0$ (say, rectangular).

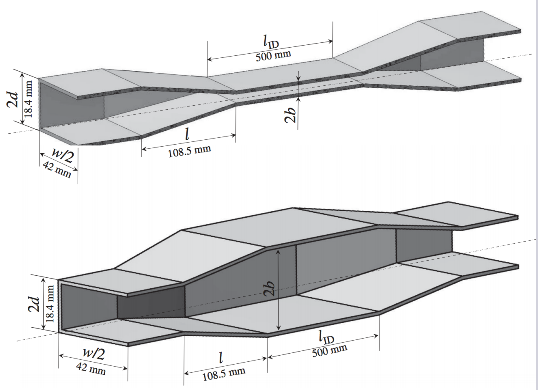
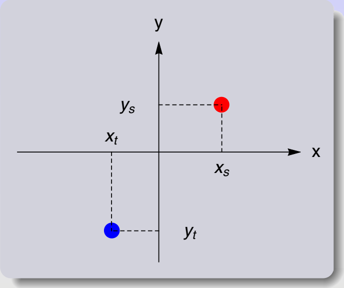


Figure from Ref.[7]: w is the width of the structure and $d(s)$ is the height.

[7] V. Smaluk et al. PRST-AB **17**, 074402 (2014).

Transverse wakes in non-axisymmetric systems



Due to the symmetry, the transverse wake is zero for zero offsets. For small offsets, in linear approximation, the transverse wake in this geometry has 4 components.

The transverse wake along y is proportional to $y_{\rm s}$ and $y_{\rm t}$

$$w_y = \bar{w}_y^d(s) y_{\rm s} + \bar{w}_y^q(s) y_{\rm t}$$

with $y_{\rm s}$ the offset (in y) of the leading (source) particle and $y_{\rm t}$ the offset of the trailing (test) one. Here $\bar{w}_y^d$ is the *dipole wake* in y direction and $\bar{w}_y^q$ is the *quadrupole wake* in y direction[8]. The bar indicates that this is the wake per unit offset. Note that, because of the symmetry, there are no terms proportional to x_s and x_t in w_y.

[8] This quadrupole should not be confused with the wake $m = 2$ in axisymmetric geometry.

Transverse wakes in non-axisymmetric systems

Similarly, the wake along x is

$$w_x(s, x_\mathrm{s}, x_\mathrm{t}) = \bar{w}_x^d(s)x_\mathrm{s} + \bar{w}_x^q(s)x_\mathrm{t}$$

Here $\bar{w}_x^d$ is the dipole wake in x direction and $\bar{w}_x^q$ is the quadrupole wake in x direction. As above, because of the symmetry, there are no terms proportional to y_s and y_t in w_x.

One can show that $\bar{w}_y^q(s) = -\bar{w}_x^q(s)$[9]. In the limit $w = \infty$ (parallel plates in x direction) it also follows that $\bar{w}_x^d(s) = -\bar{w}_x^q(s)$.

If both particles have the same offset (the offset for the whole beam), $x_\mathrm{s} = x_\mathrm{t} = x$, $y_\mathrm{s} = y_\mathrm{t} = y$, then

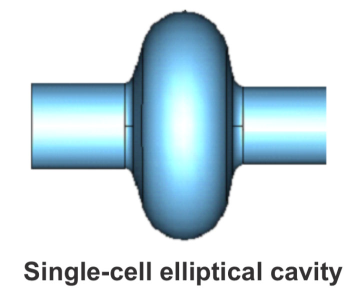
$$\begin{aligned} w_y(s, y) &= [\bar{w}_y^d(s) + \bar{w}_y^q(s)]y \\ w_x(s, x) &= [\bar{w}_x^d(s) + \bar{w}_x^q(s)]x \end{aligned} \tag{3.10}$$

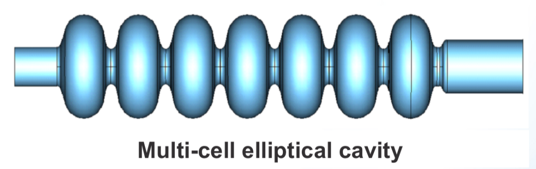
The wake in this case is the sum of the dipole and the quadrupole wakes.

[9] This follows from $\nabla^{(t)} \cdot \boldsymbol{w}_t = 0$.

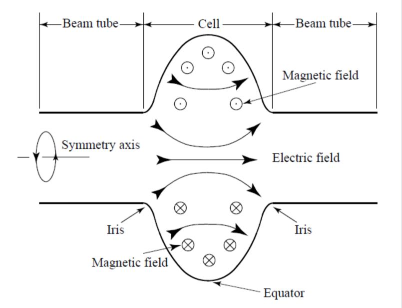
Wake field in RF cavities

Superconducting RF cavities.





Resonator wake



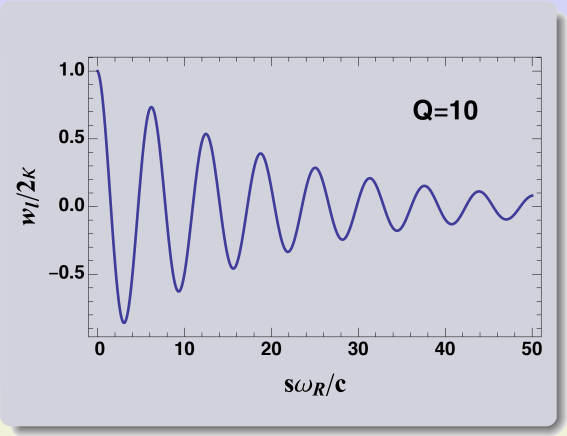
An RF cavity has many modes that are trapped inside and each oscillates with its own frequency ω_n, $n = 1, 2, \ldots$. Each mode can be considered as a damped linear oscillator, so the wake is characterized by 3 parameters: the frequency ω_R, the damping constant α and the *loss factor* $\varkappa$ (dimension V/C)

$$w_\ell(s) = 2\varkappa e^{-\alpha s/c} \left(\cos \frac{\bar{\omega} s}{c} - \frac{\alpha}{\bar{\omega}} \sin \frac{\bar{\omega} s}{c} \right) \tag{3.11}$$

The wake at the origin: $w_\ell(0) = 2\varkappa$.

The frequency $\bar{\omega}$ is related to the resonator frequency ω_R, $\bar{\omega} = \sqrt{\omega_R^2 - \alpha^2}$. Instead of α one can use the *quality factor* $Q = \omega_R / 2\alpha$.

Resonator wake



In the limit of large quality factor $Q \gg 1$ we have $\bar{\omega} \approx \omega_R$,

$$w_\ell \approx 2\varkappa e^{-\omega_R s/2Qc} \cos\frac{\omega_R s}{c}$$

Resonator wake

A quantity R/Q is often used instead of $\varkappa$

$$\frac{R}{Q} = \frac{2\varkappa}{\omega_R}$$

Here R is the *shunt impedance*.
In practice one has to sum over all cavity modes ($Q_n \gg 1$):

$$w_\ell(s) = 2 \sum_n \varkappa_n e^{-\omega_n s/2Q_n c} \cos \frac{\omega_n s}{c} \tag{3.12}$$

PEP-II copper RF cavities[10]

Frequency $\omega_R/2\pi$ (GHz)	Q	R/Q, (Ω)
0.480 (fundamental)	14218	116.358
1.003	128	0.360
1.288	222	7.000
1.584	300	3.870

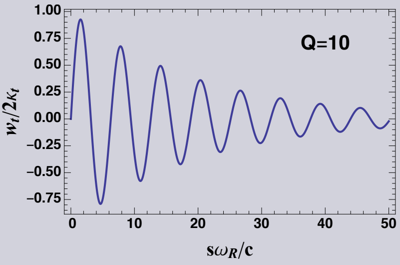
[10] Rimmer et al. SLAC-PUB-7211 (1996).

Transverse resonator wake

For an *axisymmetric* RF cavity, the transverse resonator wake (3.8) is

$$\bar{w}_t(s) = 2\sum_n \varkappa_{t,n} e^{-\omega_n s/2Q_n c} \sin\frac{\bar{\omega}_n s}{c} \tag{3.13}$$

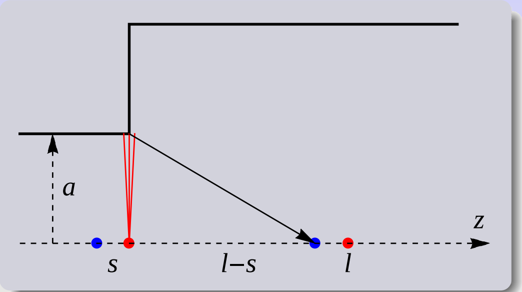
where $\varkappa_{t,n}$ are the *kick* factors (dimension V/(C m)) and the frequency $\bar{\omega}_n$ is related to the resonator frequency ω_n, $\bar{\omega}_n = \omega_n\sqrt{1-(2Q_n)^{-2}}$.



Note that $w_t(0) = 0$.

Problem: find the wake potential for the longitudinal and transverse resonant wakes.

Causality and the “catch-up” distance



In the limit when the leading charge has $v = c$, its electromagnetic field cannot overtake the charge and is localized behind it. The field can only interact with the trailing charges in the beam. This is called the *causality* principle.

Assume that a leading (driving) particle enters a cavity from a pipe of radius a at coordinate $z = 0$ at time $t = 0$. The trailing (test) particle is distance s behind the leading one. The scattered field reaches the trailing charge at time t when the leading charge is at ℓ, then $ct = \sqrt{(\ell - s)^2 + a^2}$. Assuming that $s \ll a$ we find

$$\ell = \sqrt{(\ell - s)^2 + a^2} \approx \ell \left(1 - \frac{2s}{\ell} + \frac{a^2}{\ell^2}\right)$$

$$\ell \approx \frac{a^2}{2s} \quad \text{– catch-up distance.}$$

Numerical estimate of the catch-up distance

Typically $s \sim \sigma_z$ is of order of the bunch length. Take
a=5 cm
$\sigma_z = 1$ mm

$$\ell \approx \frac{a^2}{2\sigma_z} \approx 1.25\,\mathrm{m}$$